

Lecture 18 : Poisson Processes – Part II

STAT205 Lecturer: Jim Pitman Scribe: Matias Damian Cattaneo <cattaneo@econ.berkeley.edu>

18.1 Compound Poisson Distribution

We begin by recalling some things from last lecture.

Let $X_1, X_2, \dots$ be independent and identically distributed random variables with distribution F on $\mathbb{R}$; that is:

$$F(B) = \mathbb{P}[X \in B]$$

Let N_λ be a Poisson random variable with mean λ ; that is:

$$\mathbb{P}[N_\lambda = k] = \frac{\lambda^n e^{-\lambda}}{k!}, \quad n = 0, 1, 2, \dots$$

Theorem 18.1 *Let S be the sum of a Poisson random number of i.i.d. random variables; that is:*

$$S = \sum_{i=1}^{N_\lambda} X_i.$$

Let $L(B) = \lambda F(B)$ for $B \in \mathcal{B}$, where L is a positive measure on $\mathbb{R}$. Then the distribution of S is determined by its characteristic function:

$$\mathbb{E}[e^{itS}] = \exp \left\{ \int_{\mathbb{R}} (e^{itx} - 1) L(dx) \right\}.$$

Remark: Call this distribution of S compound Poisson with parameter L (denoted by $CP(L)$) and observe that it is ∞ -divisible since:

$$CP(L) * CP(M) = CP(L + M),$$

so the convolution n -th root of $CP(L)$ is $CP(\frac{L}{n})$.

Interpretation of L : Recall that Poisson point process $\longleftrightarrow$ counting measure, and we have

$$N(B) = \sum_{i=1}^{N_\lambda} 1\{X_i \in B\}.$$

That is, $N(B)$ is the number of values $1 \leq i \leq N_\lambda$ with $X_i \in B$. Observe

$$N(\mathbb{R}) = N_\lambda \sim \text{Poisson}(\lambda)$$

What is the distribution of $N(B)$? Apply the previous theorem with X_i replaced by $1\{X_i \in B\}$. So we have

$$\mathbb{E}[e^{itN(B)}] = \exp\{(e^{it} - 1)L(B)\},$$

so $N(B) \sim \text{Poisson}(L(B))$.

More generally for $B_1, B_2, \dots, B_m$; m disjoint sets we can compute, by the same argument,

$$\mathbb{E}\left[e^{i\sum_{k=1}^m t_k N(B_k)}\right] = \prod_{k=1}^m \mathbb{E}[e^{it_k N(B_k)}],$$

and observe that the LHS is the multivariate characteristic function of the vector $(N(B_1), N(B_2), \dots, N(B_m))$ at $(t_1, t_2, \dots, t_m)$, and the RHS is the multivariate characteristic function of a collection of independent random variables with a Poisson distribution. Consequently, by the uniqueness theorem for multivariate characteristic function (see text) we conclude that $N(B_1), N(B_2), \dots, N(B_m)$ are independent Poisson variables.

18.2 Summary so far

Now we summarize our work so far.

Let $X_1, X_2, \dots$ be i.i.d. F . Let $N_\lambda \sim \text{Poisson}(\lambda)$, independent of $X_1, X_2, \dots$. Let $N(B) = \sum_{i=1}^{N_\lambda} 1\{X_i \in B\}$, the point process counting values in B up to N_λ . Then $(N(B), B \in \text{Borel})$ is a **Poisson random measure** with mean measure L , meaning that if $B_1, \dots, B_m$ are disjoint Borel sets, $(N(B_i), 1 \leq i \leq m)$ are independent with distributions $\text{Poisson}(L(B_i))$ for $1 \leq i \leq m$, respectively.

Example 18.2 (From previous lecture) Let $0 < T_1 < T_2 < \dots$ be a sum of independent $\text{Exponential}(\lambda)$ variables. So $N_t = \sum_{i=1}^{\infty} 1\{T_i \leq t\} \sim \text{Poisson}(\lambda t)$. Then we see that $(N_t, 0 \leq t \leq T)$ has the same distribution as $(N[0, t], 0 \leq t \leq T)$ where

$$N[0, t] = \sum_{i=1}^{N_\lambda} 1\{X_i \leq t\}$$

for $X_1, X_2, \dots \sim \mathcal{U}[0, T]$.

This is an example of a famous connection between sums of exponentials and uniform order statistics. Examples can be found in many texts, including [1].

These are Poisson tricks!

18.3 Computations with CP

Now we discuss some computations with $CP(L)$. Think about this: we have a Poisson scatter with mean intensity L , say $X_1, X_2, \dots, X_n$. Let $\lambda = L(\mathbb{R})$. We have

$$S = \sum_{i=1}^{N_\lambda} X_i = \int x N(dx)$$

and recall that

$$N(B) = \sum_{i=1}^{N_\lambda} 1\{X_i \in B\} \sim \text{Poisson}(L(B))$$

and also

$$\begin{aligned} N(\cdot) &= \sum_{i=1}^{N_\lambda} \delta_{X_i}(\cdot) \\ \int f(x) N(dx) &= \sum_{i=1}^{N_\lambda} f(X_i) \end{aligned}$$

Now we compute (You check details):

$$\begin{aligned} \mathbb{E}[S] &= \mathbb{E}\left[\int x N(dx)\right] = \int x \mathbb{E}[N(dx)] = \int x L(dx) \\ \mathbb{V}[S] &= \mathbb{V}\left[\int x N(dx)\right] = \mathbb{V}\left[\sum \dots\right] = \dots = \int x^2 L(dx) \end{aligned}$$

Example 18.3 Consider

$$\begin{aligned} L &= \sum_i \lambda_i \delta_{X_i} \\ N(\cdot) &= \sum_i N_i \delta_{X_i}(\cdot) \end{aligned}$$

where $N_i \sim \text{Poisson}(\lambda_i)$ and as i varies these are independent. Now we have:

$$\begin{aligned} S &= \int x N(dx) = \sum_i x_i N(x_i) \\ \mathbb{E}[S] &= \sum_i x_i \lambda_i = \int x L(dx) \\ \mathbb{V}[S] &= \sum_i x_i^2 \lambda_i = \int x^2 L(dx) \end{aligned}$$

Theorem 18.4 (L-K) *Every ∞ -divisible distribution on $\mathbb{R}$ is a weak limit of shifted CP distributions.*

Look at the characteristic function of a centered CP distribution to see something new: take $S \sim CP(L)$ and look at $(S - \mathbb{E}[S])$. Assuming that $\int |x| L(dx) < \infty$, we have

$$\begin{aligned}\mathbb{E}[e^{it(S - \mathbb{E}[S])}] &= \exp\{-it\mathbb{E}[S]\} \exp\left\{\int (e^{itx} - 1) L(dx)\right\} \\ &= \exp\left\{\int (e^{itx} - 1 - itx) L(dx)\right\}\end{aligned}$$

and

$$\mathbb{E}[(S - \mathbb{E}[S])^2] = \int x^2 L(dx)$$

from before.

Observe that this formula defines a characteristic function for every positive measure L on $\mathbb{R}$ with $L(-1, 1)^c = 0$ and $\int_{-1}^1 x^2 L(dx) < \infty$. You can easily check this; see texts such as [1]. This leads to the general L-K Formula.

18.4 More details on Lévy Measure

Definition 18.5 *A measure L on $\mathbb{R}$ is a Lévy measure if it has the following properties:*

1. $L\{(-\varepsilon, \varepsilon)^c\} < +\infty$, for all $\varepsilon > 0$.
2. $L\{0\} = 0$.
3. $\int_{-1}^1 x^2 L(dx) < +\infty$.

For such an L , $\sigma^2 \geq 0$, $c \in \mathbb{R}$, define the Lévy-Khinchine exponent in the following way:

$$\Psi_{L, \sigma^2, c}(t) = \int (e^{itx} - 1 - it\tau(x)) L(dx) - \frac{1}{2}\sigma^2 t^2 + itc,$$

where $\tau(x)$ is the truncation function defined by $\tau(x) = x\mathbf{1}_{|x| \leq 1} + \mathbf{1}_{x > 1} - \mathbf{1}_{x < -1}$.

Theorem 18.6 *Two important results:*

1. $e^{\Psi(t)}$ is an infinitely divisible characteristic function.

2. $e^{\Psi(t)}$ determines L, σ^2, c uniquely.

Before we prove this theorem, we consider a few examples.

Example 18.7 1. Consider a point mass δ_c at c . Its characteristic function is e^{itc} , and we see that $itc = \Psi_{0,0,c}(t)$.

2. Consider now a normal distribution $N(c, \sigma^2)$. Its characteristic function is $e^{itc - \sigma^2 t^2/2}$ and it is easy to see that $\Psi(t) = itc - \sigma^2 t^2/2$ corresponds to $(0, \sigma^2, c)$.

3. Now, let N be a Poisson random measure. For each $f \geq 0$, we have

$$E\left(e^{-\theta \int f dN}\right) = \exp\left(\int (e^{-\theta f(x)} - 1) \mu(dx)\right)$$

If μ is bounded measure, take $\theta = -it$,

$$E\left(e^{it \int f dN}\right) = \exp\left(\int (e^{it f(x)} - 1) \mu(dx)\right).$$

Let $L(dy) = \mu\{x : f(x) \in dy\}$ (restricted to $\{0\}^c$). For those who doesn't like to see dy 's outside the integral sign, the definition of L could be $L(B) := \mu(f^{-1}(B))$.

Then $E(e^{it \int f dN}) = \exp(\int (e^{ity} - 1) L(dy))$. Here we can recognize the enemy from the beginning of the lecture, and the characteristic function of $\int f dN$ is $\exp(\Psi_{L,0,c})$ where $c = \int \tau(x) L(dx)$.

Proof: First, we will prove that $e^{\Psi(t)}$ is a characteristic function, and the infinite divisibility is obvious (n -th root is $\Psi_{(L/n, \sigma^2/n, c/n)}$). Fix t . Observe that for $|x| < 1$ we have

$$e^{itx} - 1 - it\tau(x) = e^{itx} - 1 - itx \leq cx^2 t^2 \quad (18.1)$$

for $|xt|$ small. Therefore, the integral converges because $\int_{-1}^1 x^2 L(dx) < +\infty$ and $L\{(-\varepsilon, \varepsilon)^c\} < +\infty$. Hence $\Psi(t)$ is a well-defined complex number for all $t \in \mathbb{R}$.

Second, since the product of characteristic functions is also a characteristic function we may assume without loss of generality that σ^2 and c are both 0. Let L_n be L restricted to $(-\frac{1}{n}, \frac{1}{n})^c$. Note that $\exp(\Psi_{L_n,0,0}(t))$ is a characteristic function: since L_n is finite, $\exp(\Psi_{L_n,0,0}(t))$ is the characteristic function of a shifted compound Poisson variable with parameter L_n . From 18.1 and the dominated convergence theorem we see that

$$\lim_{n \rightarrow \infty} \Psi_{L_n,0,0}(t) = \Psi_{L,0,0}(t).$$

Since $\exp$ is continuous function we immediately have that $\exp(\Psi_{L_n,0,0}(t)) \rightarrow \exp(\Psi_{L,0,0}(t))$ and it only remains to prove that $\Psi(t)$ is continuous at 0 (in order to apply the Lévy continuity theorem). This is left as an exercise for the reader. (The same dominated convergence theorem will work.) ■

References

- [1] Richard Durrett. *Probability: theory and examples, 3rd edition*. Thomson Brooks/Cole, 2005.